

Nonhamiltonian Graphs with Given Toughness

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Abstract

In 1973, Chvátal introduced the concept of toughness τ of a graph and constructed an infinite class of nonhamiltonian graphs with $\tau = \frac{3}{2}$. Later Thomassen found nonhamiltonian graphs with $\tau > \frac{3}{2}$, and Enomoto et al. constructed nonhamiltonian graphs with $\tau = 2 - \epsilon$ for each positive ϵ . The last result in this direction is due to Bauer, Broersma and Veldman, which states that for each positive ϵ , there exists a nonhamiltonian graph with $\tau \geq \frac{9}{4} - \epsilon$. In this paper we prove that for each rational number t with $0 < t < \frac{9}{4}$, there exists a nonhamiltonian graph with $\tau = t$.

Key words: Hamilton cycle, toughness.

1 Introduction

Only finite undirected graphs without loops or multiple edges are considered. The set of vertices of a graph G is denoted by $V(G)$ and the set of edges by $E(G)$. The order and the independence number of G is denoted by n and α , respectively. For S a subset of $V(G)$, we denote by $G \setminus S$ the maximum subgraph of G with vertex set $V(G) \setminus S$. The neighborhood of a vertex $x \in V(G)$ is denoted by $N(x)$. A graph G is hamiltonian if G contains a Hamilton cycle, i.e. a cycle of length n . A good reference for any undefined terms is [5].

The concept of toughness of a graph was introduced in 1973 by Chvátal [6]. Let $\omega(G)$ denote the number of components of a graph G . A graph G is t -tough if $|S| \geq t\omega(G \setminus S)$ for every subset S of the vertex set $V(G)$ with $\omega(G \setminus S) > 1$. The toughness of G , denoted $\tau(G)$, is the maximum value of t for which G is t -tough (taking $\tau(K_n) = \infty$ for all $n \geq 1$). By the definition, toughness τ is a rational number. Since then significant progress has been made toward understanding the relationship between the toughness of a graph and its cycle structure. Much of the research on this subject have been inspired by the following conjecture due to Chvátal [6].

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Conjecture 1. There exists a finite constant t_0 such that every t_0 -tough graph is hamiltonian.

In [6], Chvátal constructed an infinite family of nonhamiltonian graphs with $\tau = \frac{3}{2}$, and then Thomassen [[4], p.132] found nonhamiltonian graphs with $\tau > \frac{3}{2}$. Later Enomoto et al. [7] have found nonhamiltonian graphs with $\tau = 2 - \epsilon$ for each positive ϵ . The last result in this direction is due to Bauer, Broersma and Veldman [2] inspired by special constructions introduced in [1] and [3].

Theorem A. For each positive $\epsilon > 0$, there exists a nonhamiltonian graph with $\frac{9}{4} - \epsilon < \tau < \frac{9}{4}$.

In view of Theorem A, the following problem seems quite reasonable.

Problem. Is there a nonhamiltonian graph G with $\tau(G) = t$ for a given rational number t with $0 < t < \frac{9}{4}$?

In this paper we prove the following.

Theorem 1. For each rational number t with $0 < t < \frac{9}{4}$, there exists a nonhamiltonian graph G with $\tau(G) = t$.

2 Preliminaries

To prove Theorem 1, we need the following graph constructions.

Definition 1. Let $L^{(1)}$ be a graph obtained from $C_8 = w_1w_2...w_8w_1$ by adding the edges w_2w_4, w_4w_6, w_6w_8 and w_2w_8 . Put $x = w_1$ and $y = w_5$. This is the well-known building block L used to obtain $(\frac{9}{4} - \epsilon)$ -tough nonhamiltonian graphs (see [2], Figure 1).

In this paper we will use a number of additional modified building blocks.

Definition 2. Let $L^{(2)}$ be the graph obtained from $L^{(1)}$ by deleting the edges w_1w_2, w_2w_8 and identifying w_2 with w_8 .

Definition 3. Let $L^{(3)}$ be the graph obtained from $L^{(1)}$ by adding a new vertex w_9 and the edges w_4w_9, w_6w_9 .

Definition 4. Let $L^{(4)}$ be the graph obtained from the triangle $w_1w_2w_3w_1$ by adding the vertices w_4, w_5 and the edges w_1w_4, w_3w_5 . Put $x = w_4$ and $y = w_5$.

Definition 5. For each $L \in \{L^{(1)}, L^{(2)}\}$, define the graph $G(L, x, y, l, m)$ ($l, m \in N$) as follows. Take m disjoint copies L_1, L_2, \dots, L_m of L , with x_i, y_i the vertices in L_i corresponding to the vertices x and y in L ($i = 1, 2, \dots, m$). Let F_m be the graph obtained from $L_1 \cup \dots \cup L_m$ by adding all possible edges between pairs of vertices in $x_1, \dots, x_m, y_1, \dots, y_m$. Let $T = K_l$ and let $G(L, x, y, l, m)$ be the join $T \vee F_m$ of T and F_m .

The following can be checked easily.

Claim 1. The vertices x and y are not connected by a Hamilton path of $L^{(i)}$ ($i = 1, 2, 3$).

The proof of the following result occurs in [1], which we repeat here for convenience.

Claim 2. Let H be a graph and x, y two vertices of H which are not connected by a Hamilton path of H . If $m \geq 2l + 1$ then $G(H, x, y, l, m)$ is non-hamiltonian.

Proof. Suppose $G(H, x, y, l, m)$ contains a Hamilton cycle C . The intersection of C and F_m consists of a collection \mathfrak{R} of at most l disjoint paths, together containing all vertices in F_m . Since $m \geq 2l + 1$, there is a subgraph H_{i_0} in F_m such that no endvertex of a path of \mathfrak{R} lies in H_{i_0} . Hence the intersection of C and H_{i_0} is a path with endvertices x_{i_0} and y_{i_0} that contains all vertices of H_{i_0} . This contradicts the fact that H_{i_0} is a copy of the graph H without a Hamilton path between x and y . Claim 1 is proved.

3 Proof of Theorem 1

Let t be a rational number with $0 < t < \frac{9}{4}$ and let $t = \frac{a}{b}$ for some integers a, b .

Case 1. $0 < \frac{a}{b} < 1$.

Let $K_{a,b}$ be the complete bipartite graph $G = (V_1, V_2; E)$ with vertex classes V_1 and V_2 of order a and b , respectively. Since $\frac{a}{b} < 1$, we have $\alpha(G) = b > (a + b)/2$ and therefore, $K_{a,b}$ is a nonhamiltonian graph. Clearly, $\tau \leq |V_1|/\omega(G \setminus V_1) = a/b$. Choose $S \subset V(G)$ such that $\tau(K_{a,b}) = |S|/\omega(G \setminus S)$. Put $S \cap V_i = S_i$ and $|S_i| = s_i$ ($i = 1, 2$). If $V_i \setminus S \neq \emptyset$ ($i = 1, 2$) then clearly $\omega(G \setminus S) = 1$, which is impossible by the definition. Hence $V_i \setminus S = \emptyset$ for some $i \in \{1, 2\}$, i.e. $V_i \subseteq S$.

Case 1.1. $i = 2$.

Since $\tau = (s_1 + b)/(a - s_1) \geq b/a$, we have $s_1 = 0$, i.e. $S = V_2$ and $\tau = b/a$, contradicting that fact that $\tau \leq a/b$.

Case 1.2. $i = 1$.

Since $\tau = (s_2 + a)/(b - s_2) \geq a/b$, we have $s_2 = 0$, implying that $S = V_1$ and $\tau = a/b$.

Case 2. $\frac{a}{b} = 1$.

Let G be a graph obtained from $C_6 = x_1x_2...x_6x_1$ by adding a new vertex x_7 and the edges x_1x_7, x_4x_7, x_2x_6 . Clearly, G is not hamiltonian and $\tau(G) = 1$.

Case 3. $1 < \frac{a}{b} < \frac{3}{2}$.

Case 3.1. $\frac{a}{b} < \frac{3}{2} - \frac{1}{b}$.

Let V_1, V_2, V_3 be pairwise disjoint sets of vertices with

$$V_1 = \{x_1, x_2, \dots, x_{a-b+1}\}, \quad V_2 = \{y_1, y_2, \dots, y_b\}, \quad V_3 = \{z_1, z_2, \dots, z_b\}.$$

Join each x_i to all the other vertices and each z_i to every other z_j as well as to the vertex y_i with the same subscript i . Call the resulting graph H . To determine the toughness of H , choose $W \subset V(H)$ such that $\tau(H) = |W|/\omega(H \setminus W)$. Put $m = |W \cap V_3|$. Clearly, W is a minimal set whose removal from H results in a graph with $\omega(H \setminus W)$ components. As W is a cutset, we have $V_1 \subset W$ and $m \geq 1$. From the minimality of W we easily conclude that $V_2 \cap W = \emptyset$ and $m \leq b - 1$. Then we have $|W| = m + a - b + 1$ and $\omega(H \setminus W) = m + 1$. Hence

$$\tau(H) = \frac{|W|}{\omega(H \setminus W)} = \min_{1 \leq m \leq b-1} \frac{m + a - b + 1}{m + 1} = \frac{a}{b}.$$

To see that H is nonhamiltonian, let us assume the contrary, i.e. let C be a Hamilton cycle in H . Denote by F the set of edges of C having at least one endvertex in V_2 . Since V_2 is independent, we have $|F| = 2|V_2|$. On the other hand, there are at most $2|V_1|$ edges in F having one endvertex in V_1 and at most $|V_3|$ edges in F having one endvertex in V_3 . Thus

$$2b = 2|V_2| = |F| \leq 2|V_1| + |V_3| = 2(a - b + 1) + b = 2a - b + 2.$$

But this is equivalent to $a/b \geq 3/2 - 1/b$, contradicting the hypothesis.

Case 3.2. $\frac{a}{b} \geq \frac{3}{2} - \frac{1}{b}$.

By choosing $q \in \mathbb{N}$ sufficiently large with

$$\frac{a}{b} = \frac{aq}{bq} < \frac{3}{2} - \frac{1}{bq},$$

we can argue as in Case 3.1.

Case 4. $\frac{a}{b} = \frac{3}{2}$.

An example of a nonhamiltonian graph with $\tau = 3/2$ is obtained when in the Petersen graph, each vertex is replaced by a triangle.

Case 5. $\frac{3}{2} < \frac{a}{b} < \frac{7}{4}$.

Claim 3. For $l \geq 2$ and $m \geq 1$,

$$\tau(G(L^{(2)}, x, y, l, m)) = \frac{l + 3m}{1 + 2m}.$$

Proof. Let $G = G(L^{(2)}, x, y, l, m)$ for some $l \geq 2$ and $m \geq 1$. Choose $S \subseteq V(G)$ such that $\omega(G \setminus S) > 1$ and $\tau(G) = |S|/\omega(G \setminus S)$. Obviously, $V(T) \subseteq S$. Define $S_i = S \cap V(L_i)$, $s_i = |S_i|$, and let ω_i be the number of components of $L_i \setminus S_i$ that contain neither x_i nor y_i ($i = 1, \dots, m$). Then

$$\tau(G) = \frac{l + \sum_{i=1}^m s_i}{c + \sum_{i=1}^m \omega_i} \geq \frac{l + \sum_{i=1}^m s_i}{1 + \sum_{i=1}^m \omega_i},$$

where $c = 0$ if $x_i, y_i \in S$ for all $i \in \{1, \dots, m\}$ and $c = 1$ otherwise. It is easy to see that

$$\omega_i \leq 2, \quad s_i \geq \frac{3}{2}\omega_i \quad (i = 1, \dots, m).$$

Then

$$\begin{aligned} \tau &\geq \frac{l + \frac{3}{2} \sum_{i=1}^m \omega_i}{1 + \sum_{i=1}^m \omega_i} = \frac{l - \frac{3}{2}}{1 + \sum_{i=1}^m \omega_i} + \frac{3}{2} \\ &\geq \frac{l - \frac{3}{2}}{1 + 2m} + \frac{3}{2} = \frac{l + 3m}{1 + 2m}. \end{aligned}$$

Set $U = V(T) \cup U_1 \cup \dots \cup U_m$, where U_i is the set of vertices of L_i having degree at least 4 in L_i ($i = 1, \dots, m$). The proof of Claim 3 is completed by observing that

$$\tau(G) \leq \frac{|U|}{\omega(G \setminus U)} = \frac{l + 3m}{2m + 1}. \quad \blacksquare$$

Case 5.1. $b = 2k + 1$ for some integer k .

Consider the graph $G(L^{(2)}, x, y, a - \frac{3}{2}(b - 1), \frac{b-1}{2})$.

Case 5.1.1. $\frac{a}{b} \leq \frac{7}{4} - \frac{9}{4b}$.

By the hypothesis,

$$m = \frac{b-1}{2} \geq 2(a - \frac{3}{2}(b-1)) + 1 = 2l + 1,$$

implying by Claim 2 that G is not hamiltonian. Clearly $b \geq 3$, implying that $m = (b-1)/2 \geq 1$.

Case 5.1.1.1. $\frac{a}{b} \geq \frac{3}{2} + \frac{1}{2b}$.

By the hypothesis, $l = a - \frac{3}{2}(b-1) \geq 2$. By Claim 3, $\tau(G) = \frac{a}{b}$.

Case 5.1.1.2. $\frac{a}{b} < \frac{3}{2} + \frac{1}{2b}$.

By choosing a sufficiently large integer q with

$$\frac{a}{b} = \frac{aq}{bq} \geq \frac{3}{2} + \frac{1}{2bq},$$

we can argue as in Case 5.1.1.1.

Case 5.1.2. $\frac{a}{b} > \frac{7}{4} - \frac{9}{4b}$.

By choosing a sufficiently large integer q with

$$\frac{a}{b} = \frac{aq}{bq} \leq \frac{7}{4} - \frac{9}{4bq},$$

we can argue as in Case 5.1.1.

Case 5.2. $b = 2k$ for some integer k .

Consider the graph G' obtained from $G(L^{(2)}, x, y, l, m)$ by replacing L_m with $L^{(3)}$.

Claim 4. For $l \geq 2$ and $m \geq 1$,

$$\tau(G') = \frac{l + 3m + 1}{2(m + 1)}.$$

Proof. Choose $S \subseteq V(G')$ such that $\omega(G' \setminus S) > 1$ and $\tau(G') = |S|/\omega(G' \setminus S)$. Obviously, $V(T) \subseteq S$. Define $S_i = S \cap V(L_i)$, $s_i = |S_i|$, and let ω_i be the number of components of $L_i \setminus S_i$ that contain neither x_i nor y_i ($i = 1, \dots, m$). Since $s_i \geq \frac{3}{2}\omega_i$ ($i = 1, \dots, m-1$) and $s_m \geq \frac{4}{3}\omega_m$, we have

$$\tau(G') \geq \frac{l + \sum_{i=1}^m s_i}{c + \sum_{i=1}^m \omega_i} \geq \frac{l + \frac{3}{2} \sum_{i=1}^{m-1} \omega_i + \frac{4}{3} \omega_m}{1 + \sum_{i=1}^m \omega_i} = \frac{l - \frac{1}{6}\omega_m}{1 + \sum_{i=1}^m \omega_i} + \frac{3}{2},$$

where $c = 0$ if $x_i, y_i \in S$ for all $i \in \{1, \dots, m\}$ and $c = 1$ otherwise. Observing also that $\omega_i \leq 2$ ($i = 1, \dots, m-1$) and $\omega_m \leq 3$, we obtain

$$(l-2) \sum_{i=1}^m \omega_i + \frac{1}{3}(m+1)\omega_m \leq (l-2)(2m+1) + (m+1) \leq 2l(m+1).$$

But this is equivalent to

$$\frac{l - \frac{1}{6}\omega_m}{1 + \sum_{i=1}^m \omega_i} + \frac{3}{2} \geq \frac{l-2}{2(m+1)} + \frac{3}{2},$$

implying that

$$\tau(G') \geq \frac{l-2}{2(m+1)} + \frac{3}{2} = \frac{l+3m+1}{2(m+1)}.$$

Set $U = V(T) \cup U_1 \cup \dots \cup U_m$, where U_i is the set of vertices of L_i having degree at least 4 in L_i ($i = 1, \dots, m$). The proof of Claim 4 is completed by observing that

$$\tau(G') \leq \frac{|U|}{\omega(G \setminus U)} = \frac{l+3m+1}{2(m+1)}. \quad \blacksquare$$

Consider the graph G' with $m = \frac{b}{2} - 1$ and $l = a - \frac{3}{2}b + 2$. Clearly $m = \frac{b}{2} - 1 \geq 1$ and $l = a - \frac{3}{2}b + 2 \geq 2$. By Claim 4, $\tau(G') = \frac{a}{b}$.

Case 5.2.1. $\frac{a}{b} \leq \frac{7}{4} - \frac{3}{b}$.

By the hypothesis, $m \geq 2l + 1$, and by Claim 2, G' is not hamiltonian.

Case 5.2.2. $\frac{a}{b} > \frac{7}{4} - \frac{3}{b}$.

By choosing a sufficiently large q with

$$\frac{a}{b} = \frac{aq}{bq} \leq \frac{7}{4} - \frac{3}{b},$$

we can argue as in Case 5.2.1.

Case 6. $\frac{7}{4} - \epsilon < \frac{a}{b} \leq 2$.

Let $m = m_1 + m_2 \geq 2l + 1$ and let G'' be the graph obtained from $G(L^{(1)}, x, y, l, m)$ by replacing L_i with $L^{(2)}$ ($i = m_1 + 1, m_1 + 2, \dots, m$). By Claim 2, G'' is not hamiltonian.

Claim 5. For $l \geq 2$, $m \geq 1$ and $m_2 \geq l - 2$,

$$\tau(G'') = \frac{l + 3m_2}{2m_2 + 1}.$$

Proof. Choose $S \subseteq V(G'')$ such that

$$\omega(G'' \setminus S) > 1, \quad \tau(G'') = |S| / \omega(G'' \setminus S).$$

Obviously, $V(T) \subseteq S$. Define $S_i = S \cap V(L_i)$, $s_i = |S_i|$, and let ω_i be the number of components of $L_i \setminus S_i$ that contain neither x_i nor y_i ($i = 1, \dots, m$). Since $s_i \geq 2\omega_i$ ($i = 1, \dots, m_1$) and $s_i \geq \frac{3}{2}\omega_i$ ($i = m_1 + 1, \dots, m$), we have

$$\begin{aligned} \tau(G'') &\geq \frac{l + \sum_{i=1}^{m_1} s_i + \sum_{i=m_1+1}^m s_i}{c + \sum_{i=1}^m \omega_i} \geq \frac{l + 2 \sum_{i=1}^{m_1} \omega_i + \frac{3}{2} \sum_{i=m_1+1}^m \omega_i}{1 + \sum_{i=1}^m \omega_i} \\ &= \frac{l + \frac{1}{2} \sum_{i=1}^{m_1} \omega_i - \frac{3}{2} + \frac{3}{2}(1 + \sum_{i=1}^m \omega_i)}{1 + \sum_{i=1}^m \omega_i} = \frac{2l + \sum_{i=1}^{m_1} \omega_i - 3}{2(1 + \sum_{i=1}^m \omega_i)} + \frac{3}{2}, \end{aligned}$$

where $c = 0$ if $x_i, y_i \in S$ for all $i \in \{1, \dots, m\}$ and $c = 1$ otherwise. Observing also that $\omega_i \leq 2$ ($i = 1, \dots, m$), we obtain

$$(2l - 3) \sum_{i=m_1+1}^m \omega_i - (2m_2 - 2l + 4) \sum_{i=1}^{m_1} \omega_i \leq 4lm_2 - 6m_2.$$

But this is equivalent to

$$\frac{2l + \sum_{i=1}^{m_1} \omega_i - 3}{2(1 + \sum_{i=1}^m \omega_i)} + \frac{3}{2} \geq \frac{2l - 3}{2(2m_2 + 1)} + \frac{3}{2},$$

implying that

$$\tau(G'') \geq \frac{2l-3}{2(2m_2+1)} + \frac{3}{2} = \frac{l+3m_2}{2m_2+1}.$$

Set $U = V(T) \cup U_1 \cup \dots \cup U_m$, where U_i is the set of vertices of L_i having degree at least 4 in L_i ($i = 1, \dots, m$). The proof of Claim 5 is completed by observing that

$$\tau(G'') \leq \frac{|U|}{\omega(G \setminus U)} = \frac{l+3m_2}{2m_2+1}. \quad \blacksquare$$

Case 6.1. $b = 2k + 1$ for some integer k .

Consider the graph G'' with $m_2 = \frac{b-1}{2}$ and $l = a - \frac{3}{2}(b-1)$.

Case 6.1.1. $\frac{a}{b} \geq \frac{3}{2} + \frac{1}{2b}$.

Since $\frac{a}{b} \leq 2$, we have

$$m_2 = \frac{b-1}{2} \geq a - \frac{3}{2}(b-1) - 2 = l - 2.$$

Next, since $\frac{a}{b} \geq \frac{3}{2} + \frac{1}{2b}$, we have $l = a - \frac{3}{2}(b-1) \geq 2$. By Claim 5, $\tau(G'') = \frac{a}{b}$.

Case 6.1.2. $\frac{a}{b} < \frac{3}{2} + \frac{1}{2b}$.

By choosing a sufficiently large integer q with

$$\frac{a}{b} = \frac{aq}{bq} \geq \frac{3}{2} + \frac{1}{2bq},$$

we can argue as in Case 6.1.1.

Case 6.2. $b = 2k$ for some integer k .

Consider the graph G''' obtained from G'' by replacing L_m with $L^{(3)}$.

Claim 6. For $l \geq 2$, $m \geq 1$ and $m_2 \geq l - 2$,

$$\tau(G''') = \frac{l+3m_2+1}{2(m_2+1)}.$$

Proof. Choose $S \subseteq V(G''')$ such that

$$\omega(G''' \setminus S) > 1, \quad \tau(G''') = |S|/\omega(G''' \setminus S)$$

Obviously, $V(T) \subseteq S$. Define $S_i = S \cap V(L_i)$, $s_i = |S_i|$, and let ω_i be the number of components of $L_i \setminus S_i$ that contain neither x_i nor y_i ($i = 1, \dots, m$). Since $s_i \geq 2\omega_i$ ($i = 1, \dots, m_1$), $s_i \geq \frac{3}{2}\omega_i$ ($i = m_1 + 1, \dots, m-1$) and $s_m \geq \frac{4}{3}\omega_m$, we have

$$\begin{aligned} \tau(G''') &\geq \frac{l + \sum_{i=1}^{m_1} s_i + \sum_{i=m_1+1}^{m-1} s_i + s_m}{c + \sum_{i=1}^m \omega_i} \\ &\geq \frac{l + 2 \sum_{i=1}^{m_1} \omega_i + \frac{3}{2} \sum_{i=m_1+1}^{m-1} \omega_i + \frac{4}{3} \omega_m}{1 + \sum_{i=1}^m \omega_i} \end{aligned}$$

$$\begin{aligned}
&= \frac{l + \frac{1}{2} \sum_{i=1}^{m_1} \omega_i - \frac{1}{6} \omega_m + (\frac{3}{2} \sum_{i=1}^{m_1} \omega_i + \frac{3}{2} \sum_{i=m_1+1}^m \omega_i)}{1 + \sum_{i=1}^m \omega_i} \\
&= \frac{l + \frac{1}{2} \sum_{i=1}^{m_1} \omega_i - \frac{1}{6} \omega_m}{1 + \sum_{i=1}^m \omega_i} + \frac{3}{2},
\end{aligned}$$

where $c = 0$ if $x_i, y_i \in S$ for all $i \in \{1, \dots, m\}$ and $c = 1$ otherwise. Observing also that $\omega_i \leq 2$ ($i = 1, \dots, m-1$) and $\omega_m \leq 3$, we obtain

$$(l-2) \sum_{i=m_1+1}^m \omega_i + \frac{1}{3}(m_2+1)\omega_m - (m_2-l+3) \sum_{i=1}^{m_1} \omega_i \leq l + 2lm_2 + 2.$$

But this is equivalent to

$$\frac{l + \frac{1}{2} \sum_{i=1}^{m_1} \omega_i - \frac{1}{6} \omega_m}{1 + \sum_{i=1}^m \omega_i} + \frac{3}{2} \geq \frac{l-2}{2(m_2+1)} + \frac{3}{2},$$

implying that

$$\tau(G''') \geq \frac{l-2}{2(m_2+1)} + \frac{3}{2} = \frac{l+3m_2+1}{2(m_2+1)}.$$

Set $U = V(T) \cup U_1 \cup \dots \cup U_m$, where U_i is the set of vertices of L_i having degree at least 4 in L_i ($i = 1, \dots, m$). The proof of Claim 6 is completed by observing that

$$\tau(G''') \leq \frac{|U|}{\omega(G \setminus U)} = \frac{l+3m_2+1}{2(m_2+1)}. \quad \blacksquare$$

Consider the graph G''' with $m_2 = \frac{b}{2} - 1$ and $l = a - \frac{3}{2}b + 2$.

Case 6.2.1. $\frac{a}{b} \leq 2 - \frac{1}{b}$.

By the hypothesis, $m_2 = \frac{b}{2} - 1 \geq (a - \frac{3}{2}b + 2) - 2 = l - 2$. Next, since $\frac{a}{b} > \frac{7}{4} - \epsilon > \frac{3}{2}$, we have $l = \frac{3}{2}b + 2 \geq 2$. By Claim 6, $\tau(G''') = \frac{a}{b}$.

Case 6.2.2. $\frac{a}{b} > 2 - \frac{1}{b}$.

By choosing a sufficiently large integer q with $\frac{a}{b} = \frac{aq}{bq} \leq 2 - \frac{1}{bq}$, we can argue as in Case 6.2.1.

Case 7. $2 < \frac{a}{b} < \frac{9}{4}$.

Case 7.1. $b = 2k + 1$ for some integer k .

Case 7.1.1. $\frac{a}{b} \leq \frac{9}{4} - \frac{11}{4b}$.

Take the graph $G(L^{(1)}, x, y, a - 2b + 2, \frac{b-1}{2})$. Since $\frac{a}{b} > 2$, we have $l = a - 2b + 2 \geq 2$. Next, the hypothesis $\frac{a}{b} \leq \frac{9}{4} - \frac{11}{4b}$ is equivalent to

$$m = \frac{b-1}{2} \geq 2(a - 2b + 2) + 1 = 2l + 1.$$

By Claim 1, $G(L^{(1)}, x, y, a - 2b + 2, \frac{b-1}{2})$ is not hamiltonian. The toughness $\tau(G(L^{(1)}, x, y, a - 2b + 2, \frac{b-1}{2}))$ can be determined exactly as in proof of Theorem A [2],

$$\tau(G(L^{(1)}, x, y, a - 2b + 2, \frac{b-1}{2})) \geq \frac{l+4m}{2m+1} = \frac{a}{b}.$$

Case 7.1.2. $\frac{a}{b} > \frac{9}{4} - \frac{11}{4b}$.

By choosing a sufficiently large integer q with

$$\frac{aq}{bq} = \frac{a}{b} \leq \frac{9}{4} - \frac{11}{4bq},$$

we can argue as in Case 7.1.1.

Case 7.2. $b = 2k$ for some positive integer k .

Take the graph G'''' obtained from $G(L^{(1)}, x, y, a - 2b + 2, \frac{b}{2})$ by replacing L_m with $L^{(4)}$. Since $\frac{a}{b} > 2$, we have $l = a - 2b + 2 > 2$. We have also $m = \frac{b}{2} > 1$, since $b \geq 3$.

Claim 7. For $l \geq 2$ and $m \geq 1$,

$$\tau(G''') = \frac{l + 4m - 2}{2m}.$$

Proof. Choose $S \subseteq V(G''')$ such that

$$\omega(G''' \setminus S) > 1, \quad \tau(G''') = |S|/\omega(G''' \setminus S)$$

Obviously, $V(T) \subseteq S$. Define $S_i = S \cap V(L_i)$, $s_i = |S_i|$, and let ω_i be the number of components of $L_i \setminus S_i$ that contain neither x_i nor y_i ($i = 1, \dots, m$). Since $s_i \geq 2\omega_i$ ($i = 1, \dots, m$), $\omega_i \leq 2$ ($i = 1, \dots, m - 1$) and $\omega_m \leq 1$, we have

$$\begin{aligned} \tau(G''') &= \frac{l + \sum_{i=1}^m s_i}{c + \sum_{i=1}^m \omega_i} \geq \frac{l + 2 \sum_{i=1}^m \omega_i}{1 + \sum_{i=1}^m \omega_i} \\ &= \frac{l - 2}{1 + \sum_{i=1}^m \omega_i} + 2 \geq \frac{l - 2}{2m} + 2 = \frac{l + 4m - 2}{2m}, \end{aligned}$$

where $c = 0$ if $x_i, y_i \in S$ for all $i \in \{1, \dots, m\}$ and $c = 1$ otherwise. Set $U = V(T) \cup U_1 \cup \dots \cup U_m$, where U_i is the set of vertices of L_i having degree at least 4 in L_i ($i = 1, \dots, m$). The proof of Claim 7 is completed by observing that

$$\tau(G''') \leq \frac{|U|}{\omega(G \setminus U)} = \frac{l + 4m - 2}{2m}. \quad \blacksquare$$

Case 7.2.1. $\frac{a}{b} \leq \frac{9}{4} - \frac{3}{b}$.

By the hypothesis,

$$m - 1 = \frac{b}{2} - 1 \geq 2(a - 2b + 2) + 1 = 2l + 1.$$

By Claim 2, G'''' is not hamiltonian. By Claim 7, $\tau(G''') = \frac{a}{b}$.

Case 7.2.2. $\frac{a}{b} > \frac{9}{4} - \frac{3}{b}$.

By choosing a sufficiently large integer q with

$$\frac{aq}{bq} = \frac{a}{b} \leq \frac{9}{4} - \frac{3}{3bq},$$

we can argue as in Case 7.2.1. Theorem 1 is proved. \blacksquare

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